

Large- N approximation for single- and two-component dilute Bose gases

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We discuss the mean-field theories obtained from the leading order in a large- N approximation for one- and two- component dilute Bose gases. For a one-component Bose gas this approximation has the following properties: the Bose-Einstein condensation (BEC) phase transition is second order but the critical temperature T_c is not shifted from the non-interacting gas value T_0 . The spectrum of excitations in the BEC phase resembles the Bogoliubov dispersion with the usual coupling constant replaced by the running coupling constant which depends on both temperature and momentum. We then study two-component Bose gases with both inter- and intra- species interactions and focus on the stability of the mixture state above T_c . Our mean-field approximation predicts an instability from the mixture state to a phase-separated state when the ratio of the inter-species interaction strength to the intra-species interaction strength (assuming equal strength for both species) exceeds a critical value. At high temperature this is a structural transition and the global translational symmetry is broken. Our work complements previous studies on the instability of the mixture phase in the presence of BEC.

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I. INTRODUCTION

With the capability of tuning inter-particle interactions and the realization of superfluid phases (see [1, 2] for reviews), cold atoms offer a window of unparalleled promises onto many-body physics. While the cold atom prospect of studying quantum criticality has attracted much attention [3], the finite temperature thermodynamics [4] offers an equally fertile ground for explorations of fundamental questions. The power of effective field theory has also proven to be useful in treating other aspects of many-body physics in cold atoms [5]. Recently developed technologies for accurate temperature determination [6] and for creating stable, flat (bulk-like) trapping potentials [7] and ring-shape potentials [8] provide some of the necessary tools for probing thermodynamics and phase transitions of cold atoms. Phase separation, the demixing phenomenon that spontaneously breaks the global translational symmetry, was the second transition after Bose-Einstein condensation (BEC) to be observed in the cold-atom laboratory [9]. The phase separation transition that provides a paradigm of second-order scaling physics in ordinary finite temperature phase transitions was observed in a mixture of dilute BECs near zero temperature. With cold atom technology, the dynamics of a zero temperature miscible-immiscible transition of bosonic superfluid mixtures, first discussed in studies motivated by the plans of creating liquid ^4He - ^6He mixtures [10], can now be studied in trapped atoms [11]. The magnetically controlled Feshbach resonance [2, 12, 13] provides a direct trigger and promises a useful probe of the interaction dependence of the phases and phase boundaries. Here we describe the finite temperature phase separation transition above the critical temperature T_c of BEC in a two-component boson mixture.

The thermodynamic descriptions of cold atoms encountered fundamental challenges posed by the inherent limitations of the gapless and conserving approximations [14] of interacting bosons. As a consequence, the treatments of the BEC-transition in a single component boson gas generally obtain a transition that is first order whereas it is known to be second order [4]. The long-standing problem of the interaction dependence of the T_c of a single component BEC (see [4] for a review) was then discussed in the low density limit by intricate reasoning tailored to the computation of T_c (only) [15, 16]. More recently [17, 18], we developed an auxiliary field description that can overcome the obstacles of the conserving/gapless approximations and reproduce the correct order of the BEC-transition at the mean-field level. This formalism, the Leading Order Auxiliary Field (LOAF) approximation, introduces two composite fields, one to describe the density $\phi^*\phi$ and one to describe the anomalous density $\phi\phi$, where ϕ denotes the bosonic field. This treatment also predicted a low density limit of the T_c -dependence upon the scattering length consistent with previous work based on large- N expansions [17] while providing a complete description from which all thermodynamic quantities can be computed [18]. Above the critical temperature, where the anomalous density vanishes, this description becomes identical to the usual large- N approximation [19]. Below, we describe a two-component Bose gas in the large- N approximation and investigate the stability of the homogeneous (mixture) phase above the BEC transition temperature to the leading order. To provide context and to gauge the performance of this method, we also derive and discuss the large- N predictions for the thermodynamics of a single-component finite-temperature Bose gas.

Although the relativistic $O(N)$ model for self-interacting bosons has been the subject of many papers [20–23], the non-relativistic version which introduces N replicas of the $O(2)$ or equivalently $U(1)$ symmetry of the non-relativistic dilute gas effective theory has not been discussed in great detail in the literature. The main use of the large- N expansion in BEC theories has been to discuss a large- N approximation near the

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critical temperature T_c as done by Baym et. al. [16] and Arnold and Tomasik [24] as well as in the work of Braaten and Radescu [15] and has been reviewed by Ref. [4]. In this paper we will discuss the broken symmetry aspects of this problem following the classic paper of Ref. [20]. For pedagogical purposes we will use in this paper a more general method of introducing auxiliary fields discussed by Refs. [19, 25] and briefly discuss the (equivalent) Hubbard-Stratonovich method that we used in [17, 18] and which was also used in [20]. The Hubbard-Stratonovich method relies on one having quartic interactions so it may be appropriate to introduce the atomic physics community to the more general method of introducing auxiliary fields that can be used for arbitrary polynomial (as well as non-polynomial) interactions that preserve the replication symmetry [26]. In this paper we will first derive the large- N expansion for a single component Bose gas. The large- N expansion when evaluated (as we do here) at $N = 1$ is equivalent to choosing $\theta = 0$ in the auxiliary-field approach discussed in detail in Ref. [18] so it does not include the anomalous density explicitly. In Refs. [17, 18] we introduced a loop counting parameter ϵ , which is identical to the loop counting parameter $1/N$ above T_c where the anomalous density vanishes.

What we will show is that for a one-species gas of bosons, the leading order in our large- N approximation leads to a non-perturbative (in coupling constant) mean-field theory with reasonable features. As in the more sophisticated LOAF treatment, the leading-order large- N approximation also predicts the correct second order BEC transition. Importantly, we will show that the large- N theory *does* lead to a Bogoliubov-like spectrum for temperatures below T_c because of mixing between the fluctuations of the boson and the composite field when there is a broken symmetry. However, a shortcoming of this expansion is that it does not predict in the leading order in $1/N$ a shift in the critical temperature from that of the free gas, a feature that is shared by the Popov approximation [18]. This defect is rectified at the mean-field level by also including an auxiliary field for the anomalous condensate as in the LOAF approximation. Since the LOAF approximation leads to the same result as the large- N approximation above T_c and we are interested in the stability of the mixture phase of a two-component Bose gas above T_c , we will study the simpler large- N approximation here, which ignores the contributions from the anomalous density in the condensate regime in the leading order. As summarized in Ref. [4] once the $1/N$ corrections (and higher order corrections) to the self energy of the boson propagator are included, one does find a shift in T_c so that the $1/N$ expansion at higher orders does include the effects of the anomalous density. Finally, we note that if comparisons with experimental results in an inhomogeneous trap are needed, one may use the local density approximation [27] to include possible inhomogeneity effects.

This paper is organized as the following. Section II presents our large- N approximation for a single-component interacting Bose gas both below and above the BEC transition temperature. The excitation spectrum in the BEC phase will be analyzed in details. Section III shows our large- N approximation for two-component Bose gases in the mixture state as well as

the phase-separated state. A phase transition between the two states is found in the normal phase and we present a phase diagram from our theory. Section IV concludes our work.

II. LARGE- N THEORY FOR A SINGLE-COMPONENT BOSE GAS

The partition function of a single-component Bose gas can be given a many-body theory path-integral representation [4, 28],

$$Z[V, \mu, \beta] = \iint \mathcal{D}\phi \mathcal{D}\phi^* e^{-S[\phi, \phi^*; V, \mu, \beta]}, \quad (1)$$

where we are using the Matsubara imaginary time formalism. $\beta = 1/(k_B T)$, μ is the chemical potential, and V is the volume of the system. The Euclidian action $S[\phi, \phi^*; V, \mu, \beta]$ is given by

$$S[\phi, \phi^*; V, \mu, \beta] = \int [dx] \mathcal{L}[\phi, \phi^*; \mu], \quad (2)$$

where we have introduced the notation

$$\int [dx] = \int d^3x \int_0^\beta d\tau. \quad (3)$$

For a dilute Bose gas the effective field theory for the problem can be describe by the Euclidian Lagrangian density [4]

$$\mathcal{L} = \frac{\hbar}{2} [\phi^* \partial_\tau \phi - \phi \partial_\tau \phi^*] - \frac{1}{2} \left[\phi^* \frac{\hbar^2 \nabla^2}{2m} \phi + \phi \frac{\hbar^2 \nabla^2}{2m} \phi^* \right] - \mu \phi^* \phi + \frac{1}{2} \lambda (\phi^* \phi)^2. \quad (4)$$

Here λ is the bare coupling constant and we will discuss its renormalization later. This Lagrangian density corresponds to the Hamiltonian $H = \int d^3x \left[-\frac{1}{2} \left(\phi^* \frac{\hbar^2 \nabla^2}{2m} \phi + \phi \frac{\hbar^2 \nabla^2}{2m} \phi^* \right) + \frac{1}{2} \lambda (\phi^* \phi)^2 \right]$. In what follows we set $\hbar = 1$ and $k_B = 1$. To determine the finite-temperature effective potential for the theory we will be interested in the generating functional for the connected correlation functions, $\ln Z[j]$, where

$$Z[V, \mu, \beta, j] = \iint \mathcal{D}\phi \mathcal{D}\phi^* e^{-S[\phi, \phi^*; V, \mu, \beta] + \int dx j^* \phi + \phi^* j}, \quad (5)$$

The large- N expansion is a combinatoric trick that reorganizes the Feynman diagrams of the theory in a non-perturbative fashion. First it sums the loops (bubbles) contributing to the scattering amplitude. Calling this bubble sum a “composite-field propagator”, one then re-sums the theory by implementing a loop expansion in terms of the number of composite-field propagator loops in a diagram. This way of organizing the Feynman diagrams of the theory can be accomplished formally by introducing N copies of the original theory which is equivalent to introducing a “color” index with N components, or in some cases extending a theory with a

$O(1)$ or $O(2)$ symmetry to an $O(N)$ symmetry. For the Lagrangian density (4) one then introduces (see below) a composite field $\alpha = \frac{\lambda}{N} \sum_n \phi_n^* \phi_n$ into the theory by introducing formally a “1” into the generating functional $Z[j]$ shown in Eq. (5) by using a functional expression for the delta function enforcing the definition of α . Equivalently the composite field can be introduced using a Hubbard-Stratonovich transformation as shown in [17, 18, 20]. As shown below this converts the quartic self interaction into a trilinear interaction that is quadratic in the original ϕ field. This allows one to perform the path integrals over the original fields ϕ_n in the generating functional $Z[j]$ exactly while keeping α fixed. At this stage one could then determine α by using its definition and obtain the Weiss self-consistent mean-field theory [29]. The beauty of the path integral approach is that this mean-field theory is the first term in a complete resummation of the theory in terms of loops of higher and higher numbers of the composite-field propagators [22]. Having N copies of the original theory or extending $O(2)$ to $O(N)$ introduces a small parameter $1/N$ into the theory which allows one to perform the remaining path integration over the composite field α , which arises from inserting the formal expression for the delta function into the path integral, using Laplace’s method (or the method of steepest descent).

To make this procedure explicit, we first make N copies of the original field [20, 28] by generalizing $\phi \rightarrow \phi_1, \dots, \phi_N$, rescale the coupling constant $\lambda \rightarrow \lambda/N$, and define $\Phi = (\phi_1, \phi_1^*, \dots, \phi_N, \phi_N^*)^T$ and add external sources J so that the generating functional for the correlation functions is given by

$$Z[J] = \int \left(\prod_{n=1}^N \mathcal{D}\phi_n \mathcal{D}\phi_n^* \right) e^{-S[J, \phi_n, \phi_n^*]}, \quad (6)$$

where the action S is given by

$$S = \int [dx] \left[\frac{1}{2} \Phi^\dagger \tilde{G}_0^{-1} \Phi + \frac{\lambda}{2N} \left(\sum_{n=1}^N \phi_n^* \phi_n \right)^2 - J^\dagger \Phi \right]. \quad (7)$$

Here $J = (j_1, j_1^*, j_2, j_2^*, \dots, j_N, j_N^*)^T$ is the source coupled to Φ , $\tilde{G}_0^{-1} = \text{diag}(h^{(+)}, h^{(-)}, \dots, h^{(+)}, h^{(-)})$ (N identical copies), $\tilde{G}_0^{-1} = \tilde{G}_0^{-1} - \text{diag}(\mu, \mu, \dots, \mu, \mu)$ is the bare (non-interacting) Green’s function, and $h^{(\pm)} = \pm \partial_\tau - \nabla^2/(2m)$. The classical value of the n -th field is $\phi_{n,c} = (1/Z)(\delta Z/\delta j_n^*)$. Details of the large- N approach and its applications to other fields can be found in Refs. [19, 30, 31]

We then introduce the auxiliary field $\alpha = \frac{\lambda}{N} \sum_{n=1}^N \phi_n^* \phi_n$ to facilitate our resummation scheme outlined above by inserting the following identity inside the path integral for the generating functional (1) using a formal integral representation of the Dirac delta function

$$\begin{aligned} 1 &= \int \mathcal{D}\alpha \delta\left(\alpha - \frac{\lambda}{N} \sum_{n=1}^N \phi_n^* \phi_n\right) \\ &= \mathcal{N} \int \mathcal{D}\chi \mathcal{D}\alpha \exp \left[\frac{N}{\lambda} \chi \left(\alpha - \frac{\lambda}{N} \sum_{n=1}^N \phi_n^* \phi_n \right) \right]. \end{aligned} \quad (8)$$

Here $\mathcal{N} = 1/(2\pi i)$ is a normalization factor and the χ integration contour runs parallel to the imaginary axis as discussed in Ref. [19]. This representation allows one to replace $\sum_{n=1}^N \phi_n^* \phi_n$ by $(N/\lambda)\alpha$ in S inside the path integral. Let $G_0^{-1} \equiv \tilde{G}_0^{-1} + \text{diag}(\chi, \chi, \dots, \chi, \chi)$. It is now possible to perform the quadratic integral over ϕ_n exactly to obtain a new effective action that (because of the large factor N) can be evaluated by Laplace’s method. After integrating out ϕ_n , one has

$$Z[J, S, K] = \int \mathcal{D}\chi \mathcal{D}\alpha e^{-S_{eff}}, \quad (9)$$

where we have added sources for the auxiliary fields χ and α and

$$\begin{aligned} S_{eff} &= \int [dx] \left[-\frac{1}{2} J^\dagger G_0 J - \frac{N}{\lambda} \mu \alpha + \frac{N}{2\lambda} \alpha^2 - \frac{N}{\lambda} \chi \alpha + \right. \\ &\quad \left. \frac{1}{2} \text{Tr} \ln G_0^{-1} - (S\chi + K\alpha) \right]. \end{aligned} \quad (10)$$

The $\text{Tr} \ln G_0^{-1}$ term comes from the Gaussian integration over the bosonic fields (see Eq. (11)). Note that the first term and the $\text{Tr} \ln G_0^{-1}$ term are just N copies of the $U(1)$ theory so they are of order N . We can also rescale the sources S and K to be proportional to N so that a large parameter N is in front of the entire action. This enables us to evaluate the remaining integrals over χ and α by Laplace’s method (or the stationary phase approximation). The resulting expansion is a loop expansion in the composite field propagators for χ and α [22].

The leading order in large- N expansion is obtained by just keeping the contribution to Z evaluated at the minimum of the effective action S_{eff} (i.e. the stationary phase contribution) [20, 21]. Note that $\delta S_{eff}/\delta j_n^* = -\phi_{n,c}$. Although we are interested in the theory with $N = 1$, for many problems the large- N expansion (which is an asymptotic expansion) gives qualitatively good results at $N = 1$ at leading order and the corrections at next order bring one closer to the exact answer. This was seen in the calculation of T_c (in a slightly different context) in Ref. [24].

Exactly the same large- N expansion can be obtained from completing the square in a shifted Gaussian integral. This is the well known Hubbard-Stratonovich transformation which is useful when the interactions are only quartic in nature and is based on the identity (here given for multi-dimensional integrals) [28]

$$\begin{aligned} \int \frac{dx_1 dx_2 \dots dx_n}{(2\pi)^{n/2}} \exp \left[-\frac{1}{2} \sum_{i,j} x_i M_{ij} x_j + \sum_i x_i j_i \right] &= \\ [\det M]^{-1/2} \exp \left[\frac{1}{2} \sum_{i,j} j_i M_{ij}^{-1} j_j \right]. \end{aligned} \quad (11)$$

On a lattice, with the substitutions $j_i \rightarrow \phi^*(i)\phi(i)$ $M_{ij} \rightarrow \delta_{ij}/\lambda$, $x_i \rightarrow \alpha(i)$ we find that the ϕ integral becomes quadratic, but we now have to be able to perform the resulting (path) integration over the composite field $\alpha(x)$, which is

again done by the stationary-phase approximation. The resulting large- N expanded effective action is the same as one obtains using the more general method of introducing the composite field α once one eliminates the Lagrange multiplier field $\chi(x)$ from the problem, as will be shown below.

From the Legendre transform of S_{eff} one obtains the generating functional of the one particle irreducible graphs, which is the grand potential $\Gamma[\phi, \chi, \alpha]$ [28, 31, 32]. Explicitly,

$$\Gamma = \int [dx] (J^\dagger \Phi_c + S\chi_c + K\alpha_c) + S_{eff}, \quad (12)$$

where now Φ_c, χ_c, α_c stand for the expectation values of Φ, χ, α . We define the effective potential for static homogeneous fields ϕ_n, χ, α as $V_{eff} = \Gamma/NV\beta$. Note that the Legendre transformation introduces the expectation values of ϕ_n and ϕ_n^* in Γ and V_{eff} via $\delta\Gamma/\delta\phi_{n,c}^* = j_n$ or, equivalently, $J = G_0^{-1}\Phi$ for the expectation values. Now that we have obtained the leading order approximation, we will set $N = 1$ so that we are addressing the real dilute gas which has an $U(1)$ symmetry. At the leading order we find :

$$V_{eff} = \frac{1}{2}\Phi^\dagger G_0^{-1}[\chi]\Phi - \frac{1}{\lambda}\mu\alpha + \frac{1}{2\lambda}\alpha^2 - \frac{1}{\lambda}\chi\alpha + \frac{1}{2}\text{Tr} \ln G_0^{-1}[\chi]. \quad (13)$$

Here we dropped the subscript c for the expectation value, $\Phi = (\phi, \phi^*)^T$ for the $N = 1$ case, and $G_0^{-1}[\chi]$ has been reduced to a 2×2 matrix which depends on χ . The next order in the $1/N$ expansion involves the Gaussian fluctuations in the auxiliary fields α, χ and will not be included here.

The broken-symmetry condition is determined from the condition that we have found the true minimum of the effective potential: $\delta V_{eff}/\delta\phi^* = 0$, which becomes $\chi\phi = 0$. This imposes the following conditions: (1) In the normal phase $\phi = 0$ and a finite χ is allowed and (2) in the broken-symmetry phase, ϕ is finite so $\chi = 0$.

After Fourier transforming and summing over the Matsubara frequencies, the last term of V_{eff} becomes $\sum_k [\omega_k/2 + (1/\beta) \ln(1 - e^{-\beta\omega_k})]$, where $\omega_k = \epsilon_k + \chi$ and $\epsilon_k = k^2/(2m)$. One can eliminate the Lagrange multiplier field χ by using the minimum condition $\delta V_{eff}/\delta\alpha = 0$. Explicitly,

$$\chi = -\mu + \alpha. \quad (14)$$

Then one obtains

$$V_{eff} = (-\mu + \alpha)\phi^*\phi - \frac{1}{2\lambda}\alpha^2 + \sum_k \left[\frac{\omega_k}{2} + \frac{1}{\beta} \ln(1 - e^{-\beta\omega_k}) \right]. \quad (15)$$

Here $\omega_k = \epsilon_k - \mu + \alpha$.

We need to renormalize the theory because Eq. (15) is ultraviolet divergent. The renormalized coupling constants can be defined from the effective potential and it is the value of the scattering amplitude at zero energy- and momentum- transfer or equivalently $1/\lambda_R = \delta^2 V_{eff}/\delta\alpha\delta\alpha = 1/\lambda +$ (finite polarization terms). The polarization terms can be shown to

vanish at zero temperature, so if we define the renormalized coupling constant at $T = 0$ then $\lambda_R = \lambda$, and one may write $\lambda = 4\pi\hbar^2 a/m$, where a is the s -wave scattering length at zero temperature.

The renormalization of the chemical potential μ can be seen more clearly if we rewrite $V_{1,eff}$ in terms of χ for the moment. The unrenormalized effective potential is given by

$$V_{eff} = V_0 + \chi\phi^*\phi - \frac{1}{2\lambda}(\chi + \mu)^2 + \sum_k \left[\frac{\epsilon_k + \chi}{2} + \frac{1}{\beta} \ln(1 - e^{-\beta\omega_k}) \right]. \quad (16)$$

Here V_0 is the unrenormalized vacuum energy. In the classical theory $-\partial V_{eff}/\partial\chi = \mu/\lambda$. So defining μ_R/λ for the quantum theory via

$$-\frac{\partial V_{eff}}{\partial\chi} = \frac{\mu_R}{\lambda} \quad (17)$$

and only keeping the infinite contributions from the quantum fluctuations, we find

$$\frac{\mu_R}{\lambda} = \frac{\mu}{\lambda} - \sum_k \frac{1}{2}. \quad (18)$$

Finally there are infinite contributions to the potential that are independent of the field values. These do not contribute to the equations of motion but can be rendered finite by defining a finite constant $V_{R,0}$ via

$$V_{R,0} - \frac{\mu_R^2}{2\lambda} = V_0 - \frac{\mu^2}{2\lambda} + \frac{1}{2} \sum_k \epsilon_k. \quad (19)$$

With our choice of renormalized parameters we obtain for the renormalized effective potential

$$V_{R,eff} = V_{R,0} + \chi\phi^*\phi - \frac{1}{2\lambda}(\chi + \mu_R)^2 + \sum_k \frac{1}{\beta} \ln(1 - e^{-\beta\omega_k}). \quad (20)$$

It is often convenient to change variables and write everything in terms of α_R . Similar to Eq. (14) we introduce $\alpha_R = \chi + \mu_R$. The renormalized effective potential density becomes

$$V_{eff} = (-\mu + \alpha)\phi^*\phi - \frac{\alpha^2}{2\lambda} + \sum_k \frac{1}{\beta} \ln(1 - e^{-\beta\omega_k}). \quad (21)$$

Here we drop the subscript R and the vacuum energy. $\omega_k = \epsilon_k - \mu + \alpha$.

In the normal phase, $\phi = 0$. The equations of state are derived from $\delta V_{eff}/\delta\alpha = 0$ and $-\delta V_{eff}/\delta\mu = \rho$. Explicitly,

$$\frac{\alpha}{\lambda} = \sum_k n(\omega_k), \quad \rho = \sum_k n(\omega_k). \quad (22)$$

Here $\omega_k = \epsilon_k - \mu + \alpha$ and $n(x) = [\exp(\beta x) - 1]^{-1}$ is the Bose distribution function. We define $k_0 = \rho^{1/3}$ and use k_0^{-1} as the unit of length. The BEC transition temperature of a non-interacting Bose gas with density ρ is $T_0 =$

$2\pi\hbar^2\rho^{2/3}/[\zeta^{2/3}(3/2)k_B m]$ and we use $k_B T_0$ as our unit of energy.

In the broken-symmetry phase, ϕ is finite and the condition $\delta V_{eff}/\delta\phi^* = 0$ requires that $\mu = \alpha$. Therefore $\omega_k = \epsilon_k$, which is the same as the dispersion of a non-interacting Bose gas. This implies that the single auxiliary field large- N theory used here does not lead to a shift in the critical temperature T_c from that of a noninteracting gas. One may estimate the shift of the critical temperature by including higher-order terms (see Refs. [4, 19] and references therein). A more sophisticated mean-field theory in the BEC phase, the LOAF theory which contains two auxiliary fields, has been studied in Refs. [17, 18] and does lead to a shift in T_c in the leading order. Since the LOAF theory leads to the same result as the large- N approximation above T_c and we are interested in studying the stability of the mixture state of a two-component Bose gas above T_c , we will confine ourselves to the simpler leading order in the large- N approximation which ignores the contributions from the anomalous density. However, as we shall see below, the large- N theory *does* lead to a Bogoliubov-like spectrum below T_c . Thus it is a qualitatively reasonable approximation even below T_c . We next define $\rho_c = \phi^*\phi$ as the condensate density in the broken-symmetry phase and consider a Bose gas of density ρ . Then $\delta V_{eff}/\delta\alpha = 0$ and $-\delta V_{eff}/\delta\mu = \rho$ give

$$\frac{\alpha}{\lambda} = \rho_c + \sum_k n(\omega_k), \quad \rho = \rho_c + \sum_k n(\omega_k). \quad (23)$$

Since $\omega_k = \epsilon_k$, the second equation implies that ρ_c/ρ as a function of T is insensitive to λ in this theory.

Figure 1 shows ρ_c , μ , α , and the minimum of V_{eff} as functions of T . One important feature is that the BEC transition temperature is fixed at T_0 because the dispersion ω_k is identical to the dispersion of non-interaction bosons regardless of the interaction strength. The transition is second order because ρ_c is continuous at $T_c = T_0$. The V_{eff} that is plotted is the value of the effective potential at the minimum where $\delta V_{eff}/\delta\phi = \delta V_{eff}/\delta\alpha = 0$. A negative value of V_{eff} corresponds to positive pressure so the system should be mechanically stable. The BEC condition requires $\alpha = \mu$, which can be verified below T_c .

Once we have found the correct ground state of this theory, it is important to calculate the propagators in this ground state. This has been addressed in detail in the relativistic $O(N)$ -model in Ref. [20] and here we will follow a similar procedure in the broken-symmetry phase. In the broken symmetry phase (BEC phase) the ϕ and χ propagators mix. To calculate the propagators in the broken symmetry phase one needs to invert the matrix inverse Green's function that is obtained from the effective action which is the generator of all one-particle irreducible graphs and which is the Legendre transform of the generating functional $(-\ln Z)$. The broken symmetry ground state is described by $\chi = 0$ and $\langle\phi\rangle = \sqrt{\rho_c} > 0$, where ρ_c is found from solving Eqs. (23)

The effective action whose static part leads to Eq. (20) is

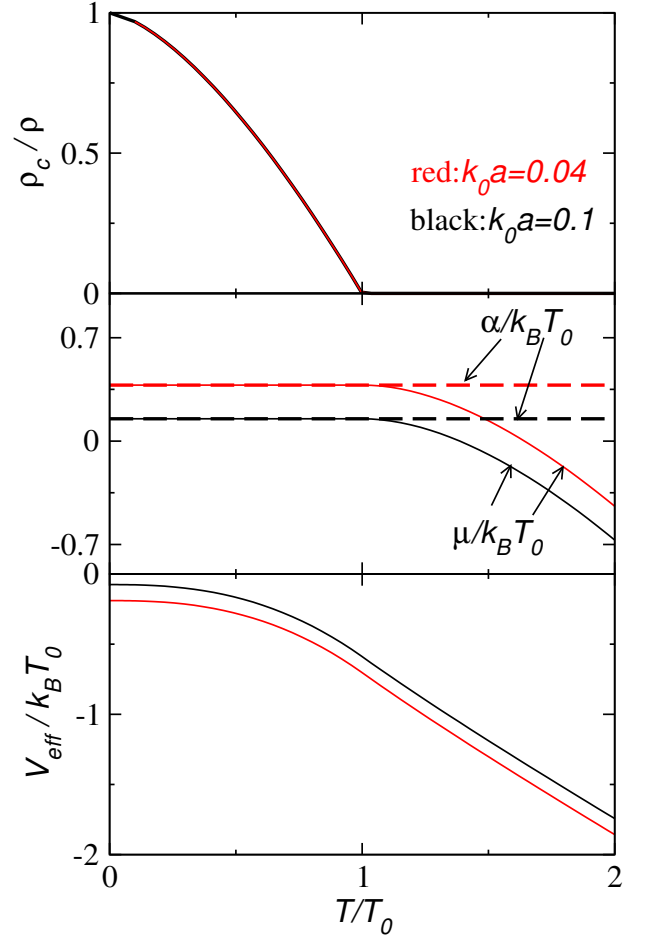


Figure 1. Single auxiliary field large- N theory of a single-component Bose gas. (a) ρ_c/ρ vs. T . (b) μ and α vs. T . (c) V_{eff} (evaluated at the minimum) as a function of T . Red and black curves correspond to $k_0 a = 0.04$ and 0.1 .

given by

$$\Gamma = \int [dx] \left(\frac{1}{2} \Phi^\dagger G_0^{-1} \Phi - \frac{N(\chi + \mu)^2}{2\lambda} + \frac{1}{2} \text{Tr} \ln G_0^{-1} \right). \quad (24)$$

Here $G_0^{-1}[\chi] = \text{diag}(-i\omega_n + \omega_k, i\omega_n + \omega_k, \dots, -i\omega_n + \omega_k, i\omega_n + \omega_k)$ and $\omega_k = \epsilon_k + \chi$. Since we are interested in the $N = 1$ case we will now set $N = 1$ and confine ourselves to the actual one-component Bose gas which has a $U(1)$, or equivalently $O(2)$, symmetry. Let us first look at the $U(1)$ approach. Here we can use the $U(1)$ symmetry of the theory to choose the vacuum expectation value of ϕ to be real. Thus in the broken symmetry phase we let $\phi = \sqrt{\rho_c} + \tilde{\phi}$, $\phi^* = \sqrt{\rho_c} + \tilde{\phi}^*$. The term $\chi\phi^*\phi$ becomes $\chi(\rho_c + \sqrt{\rho_c}\tilde{\phi} + \sqrt{\rho_c}\tilde{\phi}^* + \tilde{\phi}\tilde{\phi}^*)$. The inverse propagator in the $(\chi, \tilde{\phi})$ sector is not diagonal because of the condensate ρ_c . Let $\Psi = (\tilde{\phi}, \tilde{\phi}^*, \chi)^T$. Then the fluctuations can be written as $\Psi^\dagger \mathcal{D}^{-1} \Psi$. The inverse propagator matrix \mathcal{D}^{-1} is obtained by taking the second derivatives of the effective action and then evaluating these in the broken symmetry ground state where

$\chi = 0$ and $\langle \phi \rangle = \sqrt{\rho_c}$.

$$\mathcal{D}^{-1} = \begin{pmatrix} \frac{1}{2} \frac{\delta \Gamma_{eff}}{\delta \phi \delta \phi^*} & \frac{\delta \Gamma_{eff}}{\delta \phi \delta \phi} & \frac{1}{2} \frac{\delta \Gamma_{eff}}{\delta \phi \delta \chi} \\ \frac{\delta \Gamma_{eff}}{\delta \phi^* \delta \phi} & \frac{1}{2} \frac{\delta \Gamma_{eff}}{\delta \phi^* \delta \phi^*} & \frac{1}{2} \frac{\delta \Gamma_{eff}}{\delta \phi^* \delta \chi} \\ \frac{1}{2} \frac{\delta \Gamma_{eff}}{\delta \phi \delta \chi} & \frac{1}{2} \frac{\delta \Gamma_{eff}}{\delta \phi^* \delta \chi} & \frac{\delta \Gamma_{eff}}{\delta \chi \delta \chi} \end{pmatrix}. \quad (25)$$

The upper 2×2 submatrix is just $(1/2) \text{diag}(-i\omega_n + \omega_k, i\omega_n + \omega_k)$, $\frac{\delta \Gamma_{eff}}{\delta \phi \delta \chi} = \sqrt{\rho_c}$, and

$$\begin{aligned} \frac{\delta \Gamma_{eff}}{\delta \chi(x) \delta \chi(y)} &= -\frac{\delta(x-y)}{\lambda} - \frac{1}{2} \text{Tr} G_0 \frac{\delta G_0^{-1}}{\delta \chi(x)} G_0 \frac{\delta G_0^{-1}}{\delta \chi(y)} \\ &= -\frac{\delta(x-y)}{\lambda} - \frac{1}{2} \text{Tr} G_0(x, y) G_0(y, x). \end{aligned} \quad (26)$$

Here x, y denote the imaginary time and spatial coordinates, $\delta(x-y)$ is the four-dimensional Dirac delta function, and we have used $\delta G_0 / \delta \chi = -\int G_0 (\delta G_0^{-1} / \delta \chi) G_0$. After a Fourier transform this becomes

$$\frac{\delta \Gamma_{eff}}{\delta \chi \delta \chi}(K) = -\frac{1}{\lambda} - B(K, T). \quad (27)$$

Here $B(K, T) = (1/2) \sum_Q \text{Tr} G_0(Q) G_0(K+Q)$, $K = (i\omega_n, k)$, $Q = (i\Omega_\nu, q)$, $\sum_Q = (1/\beta) \sum_\nu \sum_q$ with ω_n and Ω_ν being bosonic Matsubara frequencies. The expression for $B(K, T)$ will be given shortly.

The inverse matrix propagator in the leading order in the broken symmetry ground state is thus

$$\mathcal{D}^{-1} = \begin{pmatrix} \frac{1}{2}(-i\omega_n + \epsilon_k) & 0 & \frac{1}{2}\sqrt{\rho_c} \\ 0 & \frac{1}{2}(i\omega_n + \epsilon_k) & \frac{1}{2}\sqrt{\rho_c} \\ \frac{1}{2}\sqrt{\rho_c} & \frac{1}{2}\sqrt{\rho_c} & -[\frac{1}{\lambda} + B(K, T)] \end{pmatrix}. \quad (28)$$

Here $\chi = 0$ in $B(K, T)$. The dispersion relation for ω is found by setting $\det \mathcal{D}^{-1} = 0$, which yields

$$\left[\frac{1}{\lambda} + B(K, T) \right] (\omega_n^2 + \epsilon_k^2) + \rho_c \epsilon_k = 0. \quad (29)$$

After the analytical continuation $i\omega_n \rightarrow \omega + i0^+$, the solution to $\det \mathcal{D}^{-1} = 0$ is

$$\omega^2 = \epsilon_k [\epsilon_k + \lambda(\omega, k, T) \rho_c]. \quad (30)$$

Here $\lambda(\omega, k, T) \equiv \lambda/[1 + \lambda B(\omega, k, T)]$ is the running coupling constant. We will show that $B(\omega, k, T = 0) = 0$ so $\lambda(\omega, k, T = 0) = \lambda$. Therefore at $T = 0$,

$$\omega^2 = \epsilon_k [\epsilon_k + \lambda \rho_c]. \quad (31)$$

One may compare this with the Bogoliubov dispersion $\omega_B^2 = \epsilon_k(\epsilon_k + 2\lambda\rho_c)$ and see that the dispersions are similar. The factor of two comes from the fact that we have ignored the contribution from the anomalous density $\langle \phi\phi \rangle$ in the lowest order of our large- N approximation. This contribution should get restored at higher order in the expansion.

These results for the inverse propagator can also be discussed in the $O(2)$ language by writing $\phi = \phi_1 + i\phi_2$ and

$\phi^* = \phi_1 - i\phi_2$. Here we use the $O(2)$ symmetry to choose the condensate in the “1” direction. Then, in the broken symmetry phase $\phi_1^{(1)} = \sqrt{\rho_c} + \sigma$ and $\phi_2^{(1)} = \pi$. The inverse propagator in the (χ, σ, π) representation takes now a slightly different form. The condensate density in this case only couples to σ but not π . We define $\bar{\Psi} = (\sigma, \pi, \chi)^T$ and the fluctuations are $\bar{\Psi}^\dagger \bar{\mathcal{D}}^{-1} \bar{\Psi}$. Using Eq. (24) we obtain

$$\bar{\mathcal{D}}^{-1} = \begin{pmatrix} \epsilon_k & -\omega_n & \sqrt{\rho_c} \\ \omega_n & \epsilon_k & 0 \\ \sqrt{\rho_c} & 0 & -[\frac{1}{\lambda} + B(K, T)] \end{pmatrix}. \quad (32)$$

Note that the time-derivative terms in Eq. (24) becomes $\pi \partial_\tau \sigma - \sigma \partial_\tau \pi$ and this results in off-diagonal elements in the sub-matrix corresponding to π and σ . From $\det \bar{\mathcal{D}}^{-1} = 0$ one finds exactly the same dispersion as the one given by Eq. (30). Therefore the Bogoliubov-like dispersion at $T = 0$ emerges when one calculates the propagators in the correct broken symmetry ground state.

Now we show $B(K, T)$ explicitly. We define $G_{11} = 1/(-i\Omega_\nu + \omega_q)$ and $G_{22} = 1/(i\Omega_\nu + \omega_q)$, where $\omega_q = \epsilon_q + \chi$. Then $B(K) = (1/2) \sum_Q [G_{11}(Q) G_{11}(Q+K) + G_{22}(Q) G_{22}(Q+K)]$. After summing over the Matsubara frequency, $B(K, T)$ becomes

$$\frac{1}{2} \sum_q \left[\frac{n(\omega_{q+k}) - n(\omega_q)}{i\omega_n + \omega_q - \omega_{q+k}} + \frac{n(\omega_q) - n(\omega_{q+k})}{i\omega_n - \omega_q + \omega_{q+k}} \right]. \quad (33)$$

Here we have used $n(\omega_q + i\omega_n) = n(\omega_q)$ and $n(x) + n(-x) = -1$. Defining $\Delta\omega \equiv \omega_{q+k} - \omega_k$ and $\Delta n \equiv n(\omega_{q+k}) - n(\omega_q)$, we obtain the result

$$B(K, T) = \frac{1}{2} \sum_q \Delta n \left[\frac{1}{i\omega_n - \Delta\omega} - \frac{1}{i\omega_n + \Delta\omega} \right]. \quad (34)$$

The function $B(\omega, k, T)$ is then evaluated by the analytic continuation $i\omega_n \rightarrow \omega + i0^+$ [33]. One important consequence follows immediately. In the broken-symmetry phase $\omega_q = \epsilon_q$ and $n(\omega_q) = 0$ as $T \rightarrow 0$. Therefore $B(\omega, k, T = 0) = 0$ and this leads to a Bogoliubov-like dispersion for the gapless mode inferred from the pole of the inverse propagator \mathcal{D}^{-1} .

At finite T one can use the identity $1/(x + i0^+) = P(1/x) - i\pi\delta(x)$, where P denotes the Cauchy principle integral, to obtain the full expression:

$$\begin{aligned} B(\omega, k, T) &= \frac{1}{2} P \sum_q \Delta n \left[\frac{1}{\omega - \Delta\omega} - \frac{1}{\omega + \Delta\omega} \right] + \\ &\quad i \frac{\pi}{2} \sum_q \Delta n [\delta(\omega + \Delta\omega) - \delta(\omega - \Delta\omega)]. \end{aligned} \quad (35)$$

Thus at finite T one has to solve Eq. (29) with $B(\omega, k, T)$ to find the (complex) dispersion.

III. THE NORMAL PHASE OF A TWO-COMPONENT BOSE GAS

The effective action of a two-component Bose gas is a generalization of Eqs. (2) and (4)

$$S[\phi_j, \phi_j^*] = \int [dx] \left\{ \sum_{j=1,2} \left[\frac{\hbar}{2} [\phi_j^* \partial_\tau \phi_j - \phi_j (\partial_\tau \phi_j^*)] - \frac{1}{2} [\phi_j^* (\frac{\nabla^2}{2m_j} \phi_j) + \phi_j (\frac{\nabla^2}{2m_j} \phi_j^*)] - \mu_j \phi_j^* \phi_j + \frac{1}{2} \lambda_j (\phi_j^* \phi_j)^2 \right] + \lambda_{12} (\phi_1^* \phi_1) (\phi_2^* \phi_2) \right\}. \quad (36)$$

We again introduce the large parameter N into the theory by the replication trick $\phi_j \rightarrow \phi_{j,n}$ where $n = 1, 2, \dots, N$ and rescale the coupling constants $\lambda_j \rightarrow \lambda_j/N$ and $\lambda_{12} \rightarrow \lambda_{12}/N$. In the following we use a similar set of symbols to denote physical quantities of a two-component Bose gas. This set of symbols should not be confused with those for a single-component Bose gas in the previous discussion.

The action with the source term after the replication becomes

$$S = \int [dx] \left[\frac{1}{2} \Phi^\dagger \tilde{G}_0^{-1} \Phi + \frac{\lambda_1}{2N} \left(\sum_{n=1}^N \phi_{1,n}^* \phi_{1,n} \right)^2 + \frac{\lambda_2}{2N} \left(\sum_{n=1}^N \phi_{2,n}^* \phi_{2,n} \right)^2 + \frac{\lambda_{12}}{N} \left(\sum_{n=1}^N \phi_{1,n}^* \phi_{1,n} \right) \times \left(\sum_{n=1}^N \phi_{2,n}^* \phi_{2,n} \right) - J^\dagger \Phi \right]. \quad (37)$$

Here we define $\Phi = (\phi_{1,1}, \phi_{1,1}^*, \phi_{2,1}, \phi_{2,1}^*, \dots)^T$, $\tilde{G}_0^{-1} = \text{diag}(h_1^{(+)}, h_1^{(-)}, h_2^{(+)}, h_2^{(-)}, \dots)$, $\tilde{G}_0^{-1} = \text{diag}(\mu_1, \mu_1, \mu_2, \mu_2, \dots)$ is the bare (noninteracting) Green's function of a two-component Bose gas, $h_j^{(\pm)} = \pm \partial_\tau - \nabla^2/(2m_j)$ for $j = 1, 2$. There are N copies in Φ , \tilde{G}_0^{-1} , and \tilde{G}_0 . J is the source coupled to Φ . The identity

$$1 = \mathcal{N}^2 \int \mathcal{D}\chi_1 \mathcal{D}\chi_2 \mathcal{D}\alpha_1 \mathcal{D}\alpha_2 \exp \left[\frac{N}{\lambda_1} \chi_1 (\alpha_1 - \frac{\lambda_1}{N} \sum_{n=1}^N \phi_{1,n}^* \phi_{1,n}) + \frac{N}{\lambda_2} \chi_2 (\alpha_2 - \frac{\lambda_2}{N} \sum_{n=1}^N \phi_{2,n}^* \phi_{2,n}) \right] \quad (38)$$

has the effect of introducing two delta functions similar to the case of a single-component Bose gas so one can replace $\sum_{n=1}^N \phi_{j,n}^* \phi_{j,n}$ by $(N/\lambda_j) \alpha_j$ in S . This replacement facilitates our resummation scheme and we will treat $1/N$ as a small parameter. Let $G_0^{-1} \equiv \tilde{G}_0^{-1} + \text{diag}(\chi_1, \chi_1, \chi_2, \chi_2, \dots)$.

After integrating out $\phi_{j,n}$, one has

$$S_{eff} = \int [dx] \left[-\frac{1}{2} J^\dagger G_0 J - \frac{N}{\lambda_1} \mu_1 \alpha_1 - \frac{N}{\lambda_2} \mu_2 \alpha_2 + \frac{N}{2\lambda_1} \alpha_1^2 + \frac{N}{2\lambda_2} \alpha_2^2 + \frac{N\lambda_{12}}{\lambda_1 \lambda_2} \alpha_1 \alpha_2 - \frac{N}{\lambda_1} \chi_1 \alpha_1 - \frac{N}{\lambda_2} \chi_2 \alpha_2 + \frac{1}{2} Tr \ln G_0^{-1} - K^\dagger X \right]. \quad (39)$$

Here $X = (\chi_1, \chi_2, \alpha_1, \alpha_2)^T$ with its source term K and expectation value X_c . As in the single-component case we evaluate the path integrals over χ_j, α_j via the method of stationary phase or steepest descent and in the leading order in large- N we keep only the contributions at the stationary phase point.

The generator of the one-particle irreducible diagrams is obtained from the Legendre transform of S_{eff} . Explicitly, $\Gamma = \int (J^\dagger \Phi_c + K^\dagger X_c + S_{eff})$, where Φ_c is the classical value of Φ . We define the effective potential as $V_{eff} = \Gamma/NV\beta$. Keeping the leading term in the $1/N$ expansion, and then setting $N = 1$ we obtain the effective potential for static homogeneous fields

$$V_{eff} = \frac{1}{2} \Phi^\dagger G_0^{-1} \Phi - \frac{1}{\lambda_1} \mu_1 \alpha_1 - \frac{1}{\lambda_2} \mu_2 \alpha_2 + \frac{1}{2\lambda_1} \alpha_1^2 + \frac{1}{2\lambda_2} \alpha_2^2 + \frac{\lambda_{12}}{\lambda_1 \lambda_2} \alpha_1 \alpha_2 - \frac{1}{\lambda_1} \chi_1 \alpha_1 - \frac{1}{\lambda_2} \chi_2 \alpha_2 + \frac{1}{2} Tr \ln G_0^{-1}. \quad (40)$$

Here $\Phi = (\phi_1, \phi_1^*, \phi_2, \phi_2^*)^T$ and G_0^{-1} has been reduced to a 4×4 matrix. Again the Legendre transformation introduces the expectation values of $\phi_{j,n}$ and $\phi_{j,n}^*$ to Γ and V_{eff} via $J = G_0^{-1} \Phi$ for the expectation values. The broken-symmetry condition is determined from the condition that we have found the true minimum of the effective potential: $\delta V_{eff}/\delta \phi_j^* = 0$, which becomes $\chi_j \phi_j = 0$. In the normal phase $\phi_j = 0$ while in the broken-symmetry phase $\chi_j = 0$. In the normal phase, the first term in V_{eff} is zero at the minimum of the potential (which occurs at $\phi_j = 0$).

In the following we will focus on the normal phase of the mixture state and consider $\rho_1 = \rho_2 = \rho_0$ and $m_1 = m_2 = m$, where ρ_0 is the density of a non-interacting single-component Bose gas with the BEC transition temperature $T_0 = 2\pi\hbar^2 \rho_0^{2/3}/[\zeta(2/3)(3/2)k_B m]$. Similar to the case of a single-component Bose gas, we define $k_0 = \rho_0^{1/3}$ and use k_0^{-1} and $k_B T_0$ as the units of length and energy.

The last term in V_{eff} can be evaluated using the standard Matsubara frequency summation technique and it becomes $\sum_{k,j} [\omega_j/2 + (1/\beta) \ln(1 - e^{-\beta\omega_j})]$, where $\omega_j = \epsilon_j + \chi_j$ and $\epsilon_j = \hbar^2 k^2/(2m_j)$. To express V_{eff} as a functional of α_j and μ_j , we use $\delta V_{eff}/\delta \alpha_j = 0$ to obtain

$$\chi_j = -\mu_j + \alpha_j + \frac{\lambda_{12}}{\lambda_j} \alpha_{\bar{j}}. \quad (41)$$

Here $\bar{j} = 1$ if $j = 2$ and $\bar{j} = 2$ if $j = 1$. This leads to

$$V_{eff} = \sum_j (-\mu_j + \alpha_j + \frac{\lambda_{12}}{\lambda_{\bar{j}}} \alpha_{\bar{j}}) \phi_j^* \phi_j - \frac{1}{2\lambda_1} \alpha_1^2 - \frac{1}{2\lambda_2} \alpha_2^2 - \frac{\lambda_{12}}{\lambda_1 \lambda_2} \alpha_1 \alpha_2 + \sum_{k,j} \left[\frac{\omega_j}{2} + \frac{1}{\beta} \ln(1 - e^{-\beta \omega_j}) \right]. \quad (42)$$

The renormalization of V_{eff} is similar to the procedure of a single-component Bose gas. Firstly one can show that $\delta^2 V_{eff} / \delta \alpha_j \delta \alpha_{\bar{j}} = -1/\lambda_j + (\text{finite terms})$ and $\delta^2 V_{eff} / \delta \alpha_1 \delta \alpha_2 = -\lambda_{12}/\lambda_1 \lambda_2 + (\text{finite terms})$. This implies that the physical coupling constants only get finite renormalization, and as in the single field case λ_j and λ_{12} are equal to their renormalized values at $T = 0$. Then one may let $\lambda_j = 4\pi\hbar^2 a_j/m_j$ and $\lambda_{12} = 2\pi\hbar^2 a_{12}/m_r$, where a_1, a_2, a_{12} are the s -wave scattering lengths of the intra- and inter-species collisions at $T = 0$ and $m_r = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. To render the theory finite, we only need to consider the (infinite) renormalization of the chemical potential and vacuum energy.

To make this procedure more transparent, we use Eq. (41) to express V_{eff} in terms of χ_j for the moment. This gives

$$V_{eff} = V_0 + \sum_j \chi_j \phi_j^* \phi_j + \frac{\lambda_2}{2\lambda^2} (\chi_1 + \mu_1)^2 + \frac{\lambda_1}{2\lambda^2} (\chi_2 + \mu_2)^2 - \frac{\lambda_{12}}{\lambda^2} (\chi_1 + \mu_1)(\chi_2 + \mu_2) + \sum_{k,j} \left[\frac{\epsilon_j + \chi_j}{2} + \frac{1}{\beta} \ln(1 - e^{-\beta \omega_j}) \right]. \quad (43)$$

Here $\bar{\lambda}^2 \equiv \lambda_{12}^2 - \lambda_1 \lambda_2$ and $\omega_j = \epsilon_j + \chi_j$. The renormalization of μ_j follows the set of equations

$$\begin{aligned} \frac{\lambda_2}{\bar{\lambda}^2} \mu_1 - \frac{\lambda_{12}}{\bar{\lambda}^2} \mu_2 + \sum_k \frac{1}{2} &= \frac{\lambda_2}{\bar{\lambda}^2} \mu_{1R} - \frac{\lambda_{12}}{\bar{\lambda}^2} \mu_{2R}, \\ \frac{\lambda_1}{\bar{\lambda}^2} \mu_2 - \frac{\lambda_{12}}{\bar{\lambda}^2} \mu_1 + \sum_k \frac{1}{2} &= \frac{\lambda_1}{\bar{\lambda}^2} \mu_{2R} - \frac{\lambda_{12}}{\bar{\lambda}^2} \mu_{1R}. \end{aligned} \quad (44)$$

This renormalization absorbs the divergent term $\sum_{k,j} (\chi_j/2)$ in V_{eff} . Then the vacuum energy is renormalized by

$$\begin{aligned} V_0 + \frac{\lambda_2}{\bar{\lambda}^2} \mu_1^2 + \frac{\lambda_1}{\bar{\lambda}^2} \mu_2^2 - \frac{\lambda_{12}}{\bar{\lambda}^2} \mu_1 \mu_2 + \sum_{k,j} \frac{\epsilon_j}{2} &= \\ V_{0R} + \frac{\lambda_2}{\bar{\lambda}^2} \mu_{1R}^2 + \frac{\lambda_1}{\bar{\lambda}^2} \mu_{2R}^2 - \frac{\lambda_{12}}{\bar{\lambda}^2} \mu_{1R} \mu_{2R}. \end{aligned} \quad (45)$$

This absorbs the divergent term $\sum_{k,j} (\epsilon_j/2)$ so there is no divergence in V_{eff} after the renormalization.

Following Eq. (41) we let $\chi_j = -\mu_{jR} + \alpha_{jR} + (\lambda_{12}/\lambda_{\bar{j}}) \alpha_{\bar{j}R}$ and rewrite V_{eff} in terms of α_{jR} . The renormalized V_{eff} is

$$V_{eff} = \sum_j (-\mu_j + \alpha_j + \frac{\lambda_{12}}{\lambda_{\bar{j}}} \alpha_{\bar{j}}) \phi_j^* \phi_j - \frac{1}{2\lambda_1} \alpha_1^2 - \frac{1}{2\lambda_2} \alpha_2^2 - \frac{\lambda_{12}}{\lambda_1 \lambda_2} \alpha_1 \alpha_2 + \sum_{k,j} \frac{1}{\beta} \ln(1 - e^{-\beta \omega_j}). \quad (46)$$

Here we drop the subscript R and the vacuum energy. We first consider the normal phase of the mixture state. From $\delta V_{eff} / \delta \alpha_j = 0$ and $\rho_j = -\delta V_{eff} / \delta \mu_j$ we obtain

$$\begin{aligned} \frac{1}{\lambda_j} \alpha_j + \frac{\lambda_{12}}{\lambda_1 \lambda_2} \alpha_{\bar{j}} &= \sum_k \left[n(\omega_j) + \frac{\lambda_{12}}{\lambda_j} n(\omega_{\bar{j}}) \right], \\ \rho_j &= \sum_k n(\omega_j). \end{aligned} \quad (47)$$

The solution along with $\phi_j = 0$ then determines the extremum of V_{eff} . To determine the stability of the mixture state, we compare the results with those obtained from the phase separated state. Since our formalism uses the grand-canonical ensemble, one has to compare different states with the same chemical potential μ_j . The state with lower V_{eff} should be energetically stable. When the two curves of V_{eff} intersect, it signals a phase transition into a different state.

The broken-symmetry phase emerges when χ_j vanishes according to the condition $\chi_j \phi_j = 0$. By analyzing (47) with $\chi_j = -\mu_j + \alpha_j + (\lambda_{12}/\lambda_{\bar{j}}) \alpha_{\bar{j}}$ one can see that this condition determines the critical temperature $T_{c,j}^{mix}$ and for each component it coincides with the BEC transition temperature of a non-interacting Bose gas with the same density. For the case $\rho_1 = \rho_2 = \rho_0$ and $m_1 = m_2 = m$, $T_{c,1}^{mix} = T_{c,2}^{mix} = T_0$, which is independent of $\lambda_1, \lambda_2, \lambda_{12}$. One has to include higher order corrections in the large- N theory to get corrections to the transition temperature.

The effective potential and equations of state for the phase-separated state are similar to those of the single-component Bose gas. For one of the species occupying part of the space, its effective potential is

$$V_{eff}^{ps} = (-\mu_j + \alpha_j^{ps})(\phi_j^{ps})^* \phi_j^{ps} - \frac{1}{2\lambda_j} (\alpha_j^{ps})^2 + \sum_k \frac{1}{\beta} \ln(1 - e^{-\beta \omega_j^{ps}}). \quad (48)$$

Here μ_j needs to match the chemical potential of species j in the mixture phase. As a consequence, the density of the phase-separated state will be different from ρ_j so we denote it by ρ_j^{ps} . The energy dispersion is $\omega_j^{ps} = \epsilon_j - \mu_j + \alpha_j^{ps}$. Since the BEC transition temperature scales as $(\rho_j^{ps})^{2/3}$, it is possible that in order to match μ_j , the phase-separated state may enter the broken-symmetry phase. Therefore we show the equations of state of the phase-separated state in the normal phase as well as in the broken-symmetry phase.

In the normal phase, $\phi_j = 0$ and

$$\frac{\alpha_j^{ps}}{\lambda_j} = \sum_k n(\omega_j^{ps}), \quad \rho_j^{ps} = \sum_k n(\omega_j^{ps}). \quad (49)$$

Here $\omega_j^{ps} = \epsilon_j - \mu_j + \alpha_j^{ps}$. In the broken-symmetry phase, $\rho_{c,j}^{ps} \equiv (\phi_j^{ps})^* \phi_j^{ps}$ and one has

$$\begin{aligned} \frac{\alpha_j^{ps}}{\lambda_j} &= \rho_{c,j}^{ps} + \sum_k n(\omega_j^{ps}), \\ \rho_j^{ps} &= \rho_{c,j}^{ps} + \sum_k n(\omega_j^{ps}). \end{aligned} \quad (50)$$

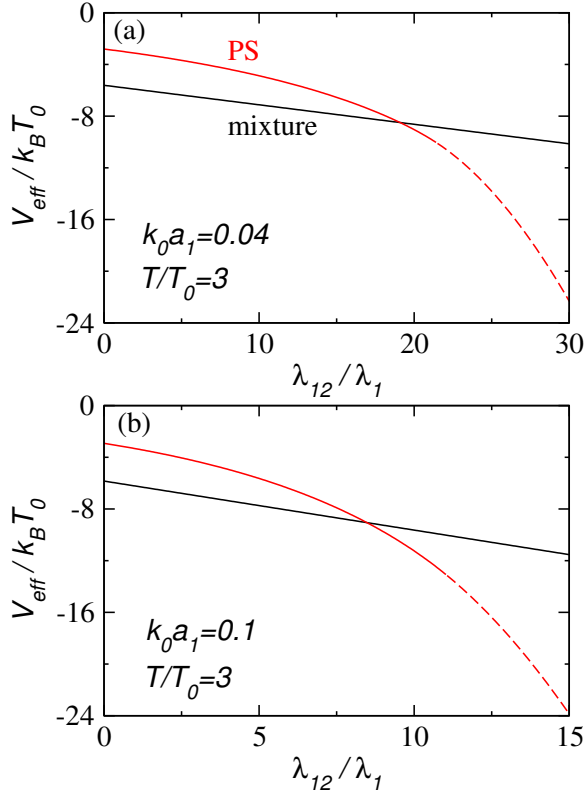


Figure 2. V_{eff} as a function of λ_{12}/λ_1 at $T/T_0 = 3$ (evaluated at the minimum) for (a) $k_0 a_1 = 0.04$ and (b) $k_0 a_1 = 0.1$. Black (red) lines corresponding to the mixture state (phase-separated state denoted by PS). The dashed lines in the phase-separated state indicates that it is in the broken-symmetry phase.

The dispersion is $\omega_j^{ps} = \epsilon_j$ due to the broken-symmetry condition $\chi_j = -\mu_j + \alpha_j = 0$.

We now focus on the case where $\lambda_1 = \lambda_2 = 4\pi\hbar^2 a_1/m$. Figure 2 shows V_{eff} from the mixture state and phase-separated state at $T/T_0 = 3$ for two selected intra-species interaction strengths $k_0 a_1 = 0.04$ and 0.1 . For small λ_{12}/λ_1 the mixture phase is more stable due to its lower V_{eff} . As λ_{12}/λ_1 reaches a critical value, the two curves of V_{eff} intersect and above the critical point the phase-separated state is more energetically stable. In the grand-canonical ensemble implemented here, the two states are compared at the *same chemical potentials*. Therefore the densities may not be the same in the two states. Note that when λ_{12}/λ_1 gets larger, the density in the phase-separated state increases in order to match the chemical potentials in the mixture state. Since there is no shift in T_c from the leading-order single-auxiliary-field theory when compared to a noninteracting Bose gas, the critical temperature $T_c^{ps} = 2\pi\hbar^2(\rho^{ps})^{2/3}/[\zeta^{2/3}(3/2)k_B m]$ of the phase-separated state increases accordingly. Eventually the phase-separated state may enter the broken-symmetry phase if T is not too high and we show this effect as the dashed lines in Fig. 2.

By locating the critical value of λ_{12}/λ_1 where the two curves of V_{eff} intersect at fixed T , we found the phase diagram shown in Figure 3. Each curve corresponds to the crit-

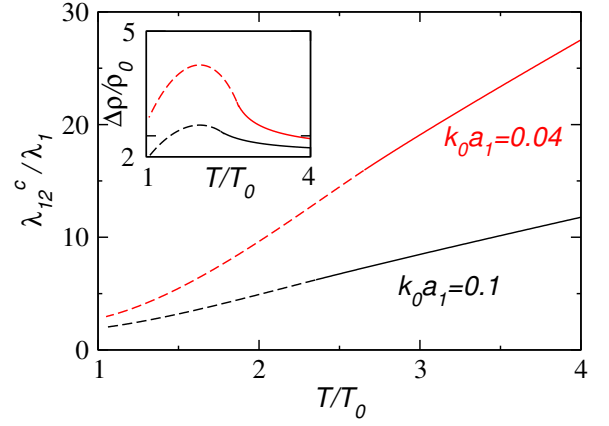


Figure 3. Phase diagram of the normal phase of a two-component Bose gas. The lines show the critical value of λ_{12}/λ_1 where a phase transition occurs for $k_0 a_1 = 0.1$ (black) and 0.04 (red). Below the critical line the system is a mixture of normal gases and above the line a phase-separated state emerges. The dashed lines indicate that the phase-separated state is in a symmetry-broken phase. Inset: The normalized density difference $(\rho_{ps} - \rho)/\rho_0$ at the critical line.

ical line separating the mixture state and the phase-separated state. One can see that the mixture state prefers lower λ_{12}/λ_1 and higher T while the phase-separated state prefers the opposite. The dashed lines in Fig. 3 indicates that the phase-separated state enters the broken-symmetry phase. In that regime the leading order in large- N approximation is only a qualitatively accurate result. In the region near and below T_c one can use the LOAF theory [17, 18] to improve on the result presented here since that approximation exactly reproduces Bogoliubov's results at weak coupling as well as correctly includes the anomalous density and predicts a shift in T_c from the free-gas result. However, using LOAF will not change the answer in the region when both states are in the normal phase and would unnecessarily complicate the simplicity of the calculation presented here. One could also include the next to leading order $1/N$ terms to be able to access the regime around T_c where the anomalous density correlations become important. A structural transition from a homogeneous mixture state into a phase-separated state in the normal phase has also been studied in two-component Fermi gases with population imbalance [34]. The underlying mechanisms are different: For fermions the system is maximizing the pairing energy while for bosons the system is minimizing the repulsive interactions.

Importantly, only the global translational symmetry is broken in the phase transition from the mixture phase to the phase-separated phase when both species are in the normal phase. In the phase-separated phase, there is an interface separating the two components and each component respects the local translational symmetry away from the interface. The different densities of the two states across this mixture to phase-separation transition remind us of the liquid-vapor transition, where no symmetry is broken and the density difference serves as the "order parameter" distinguishing the two phases. We thus study the normalized density difference $(\rho_j^{ps} - \rho_j)/\rho_0$

at the critical line in the inset of Fig. 3. If the particle number is conserved when one compares the two states, the normalized density difference should be 1 because the density in the phase-separated state should be twice as large as that of the mixture state. However, since we are comparing the two states *at the same chemical potential*, one can see from the inset of Fig. 3 that the conservation of the particle number is not respected.

To draw the phase diagram with particle number conservation, one has to work in the canonical ensemble with fixed particle numbers and find the corresponding free energy. The physics should be the same if the results are compared correctly. For an isolated atomic cloud, our instability analysis may apply to a small region with the rest of the cloud treated as a reservoir. The phase separation could start growing if the mixture state is unstable in that focused region and the instability may propagate to the whole cloud.

One has seen that the instability of a mixture of two-component Bose gases can be analyzed using the mean-field approximation derived from the leading order in the large- N expansion, which involves the introduction of an auxiliary field related to the normal density. The detailed structures which develop when the system evolves into a phase-separated state, however, require numerical simulations of the equations resulting from the effective action and are beyond

the scope of the present paper. The width of the interface separating the two species may be estimated using a variational method related to the one used in the estimation of the width of the interface separating two BEC phases in the ground state discussed in Ref. [11].

IV. CONCLUSION

We have shown that the leading order in our large- N approximation, which utilizes a single auxiliary field related to the normal density, leads to a mean-field theory usable at all couplings and temperatures that is a valuable tool for investigating the physics of interacting Bose gases. For a single-component Bose gas we show that, by constructing the propagators in the broken symmetry vacuum, a Bogoliubov-like dispersion indeed emerges. For a two-component Bose gases, this approximation predicts a normal-phase structural phase transition between a mixture state and a phase-separated state. One possible application of two-component Bose gases is to simulate cosmological dynamics [35]. Our theory may help extend this application beyond the low-temperature regime.

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